# THE CONTINUED FRACTIONS LADDER OF $(\sqrt[3]{m}, \sqrt[3]{m^2})$

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ABSTRACT. Quadratic irrationals posses a periodic continued fraction expansion. Much less is known about cubic irrationals. We do not even know if the partial quotients are bounded, even though extensive computations suggest they might follow Kuzmin's probability law. Results are given for sequences of partial quotients of $\sqrt[3]{m}$ and $\sqrt[3]{m^2}$ with $m$ noncube. A big partial quotient in one sequence finds a connection in the other.

## 1. INTRODUCTION

Several authors have considered simultaneous rational approximations to pairs of irrationals $(\alpha, \beta)$ [1, 4, 11] and new generalized concepts were developed [3]. Here however we observe a different parallelism for the specific pair $(\sqrt[3]{m}, \sqrt[3]{m^2})$ and the usual continued fraction expansion. Our long term goal is the proof of the

**Hypothesis**. The partial quotients in the continued fraction expansion of $\sqrt[3]{2}$ are unbounded.

Already in [6] this question is asked for algebraic numbers of degrees higher than 2.

A starting point for our present observation could be the long tables of partial quotients for $(\sqrt[3]{2}, \sqrt[3]{4})$ (among other algebraic irrationals) in [7], where the bigger ones are singled out. In Figure 1 we notice parallel apparition of big partial quotients in both the continued fraction expansion sequences.

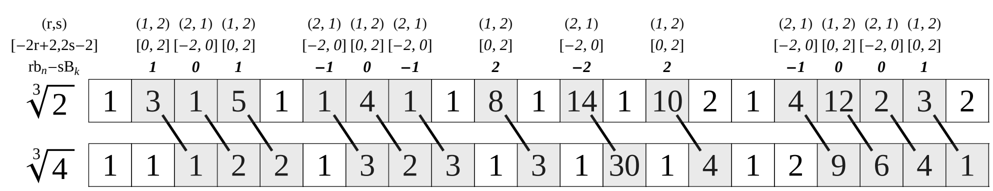


FIGURE 1. Continued fraction ladder of $(\sqrt[3]{2}, \sqrt[3]{4})$

A relatively big partial quotient in one sequence is connected to roughly half that quotient in the other sequence. These relations are formalized and analyzed in the sequel. "Big" here can be as small as 2.

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**Remarks 1.1.** Numerous numerical experiments have been carried out to support the Kuzmin statistics, giving the probability of a certain partial quotient at $P(b_n = k) = \log_2 \frac{(k+1)^2}{k(k+2)}$ [10, 7, 2].

It is also of interest that cubic irrationals appear in physics in studying chaotic and quasiperiodic motions [8, 5].

Let $\xi$ be any real number and we start the continued fraction process by saying $\xi_0 = \xi$, taking integer part $b_0 = \lfloor \xi \rfloor$ and setting

$$\xi_1 = \frac{1}{\xi_0 - b_0}, \quad b_1 = \lfloor \xi_1 \rfloor$$

only to continue as long as we can in the same fashion

$$\xi_n = \frac{1}{\xi_{n-1} - b_{n-1}}, \quad b_n = \lfloor \xi_n \rfloor. \tag{1}$$

The process eventually stops for a rational $\xi = \frac{a}{b}$ and continues indefinitely for irrational one.

We reap the approximations, convergents $\frac{p_n}{q_n}$ to $\xi$, defined by

$$p_{-1} = 1,\ q_{-1} = 0,\ p_0 = b_0,\ q_0 = 1$$

$$p_n = b_n p_{n-1} + p_{n-2}, \quad q_n = b_n q_{n-1} + q_{n-2}$$

We shall need some elementary results from [9] or [6] and we quote them here.

- Let us have an irrational $\xi = \xi_0$ and the following complete quotients are denoted by $\xi_n$, the corresponding convergents by $\frac{p_n}{q_n}$, then we can express:

$$\xi_n = -\frac{p_{n-2} - q_{n-2}\xi}{p_{n-1} - q_{n-1}\xi}. \tag{2}$$

- Even convergents are smaller than the irrational and odd ones are greater

$$\frac{p_{2j}}{q_{2j}} < \xi < \frac{p_{2j+1}}{q_{2j+1}} \tag{3}$$

  and

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} \tag{4}$$

  yielding the fact, that the convergents are in their lowest terms.
- $\frac{p_n}{q_n} = \xi + (-1)^{n-1}\frac{|\delta|}{q_n^2}$ with $|\delta| < 1$ and if $b = b_{n+1}$ be the next partial quotient there is an estimate $\frac{1}{b+2} < |\delta| < \frac{1}{b}$.
- For any rational $\frac{p}{q}$ there is a sufficient condition $|\delta| < \frac{1}{2}$ to be one of the convergents where $\delta = (\frac{p}{q} - \xi)q^2$.
- Let $\frac{p_n}{q_n}$ and $\frac{p_N}{q_N}$ be two convergents of $\xi$. Then the following statements are equivalent
  * $n < N$
  * $p_n < p_N$
  * $q_n < Q_N$
  * $|\frac{p_n}{q_n} - \xi| > |\frac{p_N}{q_N} - \xi|$

  The last inequality is a consequence of (2).

- The sequence of "relative errors" $|1-\frac{\xi}{p_n/q_n}|$ is decreasing.

$$1<|\xi_n| \overset{(2)}{=} \left|-\frac{p_{n-2}-q_{n-2}\xi}{p_{n-1}-q_{n-1}\xi}\right| = \frac{\left|1-\frac{\xi}{p_{n-2}/q_{n-2}}\right| p_{n-2}}{\left|1-\frac{\xi}{p_{n-1}/q_{n-1}}\right| p_{n-1}} < \frac{\left|1-\frac{\xi}{p_{n-2}/q_{n-2}}\right|}{\left|1-\frac{\xi}{p_{n-1}/q_{n-1}}\right|}$$

Here all fractions are in their lowest terms.

## 2. Relations in the sequences of convergents

Let $m$ be a positive integer, noncube. Then we can express the cubic roots $\xi=\sqrt[3]{m}$ and $\eta=\sqrt[3]{m^2}$ as infinite continued fractions. The triplets $(\frac{p_{n-1}}{q_{n-1}},\xi_n,b_n)$ and $(\frac{P_{k-1}}{Q_{k-1}},\eta_k,B_k)$ denotes convergents, complete quotients and partial quotients respectively.

**Definition 2.1.** We say that the triplets $(\frac{p_{n-1}}{q_{n-1}},\xi_n,b_n)$ and $(\frac{P_{k-1}}{Q_{k-1}},\eta_k,B_k)$ are *connected*, if

$$\frac{p_{n-1}}{q_{n-1}}\cdot\frac{P_{k-1}}{Q_{k-1}}=m.$$

We call the triplets together with connections *ladder* of ($\sqrt[3]{m}$, $\sqrt[3]{m^2}$) (see Figure 1 and 2).

We will need the following identities

**Lemma 2.2.**

$$\begin{aligned} m\frac{q}{p}-\sqrt[3]{m^2} &= -\frac{q}{p}\sqrt[3]{m^2}\left(\frac{p}{q}-\sqrt[3]{m}\right)\\ \left(m\frac{q}{p}-\sqrt[3]{m^2}\right)p^2 &= -\frac{p}{q}\sqrt[3]{m^2}\left(\frac{p}{q}-\sqrt[3]{m}\right)q^2\end{aligned}$$

**Proposition 2.3.** *Two different connections do not intersect.*

*Proof.* Let take two connections with convergents $\frac{p}{q}$, $\frac{p'}{q'}$ to $\sqrt[3]{m}$ such that $|\frac{p}{q}-\sqrt[3]{m}|>|\frac{p'}{q'}-\sqrt[3]{m}|$. We want to prove that $|m\frac{q}{p}-\sqrt[3]{m^2}|>|m\frac{q'}{p'}-\sqrt[3]{m^2}|$.

Using Lemma 2.2 and decreasing property of relative errors we get

$$\frac{\left|m\frac{q}{p}-\sqrt[3]{m^2}\right|}{\left|m\frac{q'}{p'}-\sqrt[3]{m^2}\right|}=\frac{\frac{q}{p}\sqrt[3]{m^2}\left|\frac{p}{q}-\sqrt[3]{m}\right|}{\frac{q'}{p'}\sqrt[3]{m^2}\left|\frac{p'}{q'}-\sqrt[3]{m}\right|}=\frac{\left|1-\frac{\sqrt[3]{m}}{p/q}\right|}{\left|1-\frac{\sqrt[3]{m}}{p'/q'}\right|}>1.$$

□

**Proposition 2.4.** *Let $\frac{p}{q}$ be convergent to $\sqrt[3]{m}$ or $\sqrt[3]{m^2}$ with partial quotient $b\geq 2m+1$. Then we have the connection to $m\frac{q}{p}$.*

*Proof.* Let $\frac{p}{q}$ be convergent to $\sqrt[3]{m}$. We have to see that $m\frac{q}{p}$ is convergent to $\sqrt[3]{m^2}$. Let $\frac{q_1}{p_1}$ be $m\frac{q}{p}$ in its lowest terms.

$$\begin{aligned}
\left|\left(\frac{q_1}{p_1}-\sqrt[3]{m^2}\right)p_1^2\right| &\leq \left|\left(m\frac{q}{p}-\sqrt[3]{m^2}\right)p^2\right| \leq \frac{p}{q}\sqrt[3]{m^2}\left|\frac{p}{q}-\sqrt[3]{m}\right|q^2 \\
&\leq \left(\sqrt[3]{m}+\frac{1}{bq^2}\right)\sqrt[3]{m^2}\frac{1}{b} < \left(m+\frac{m}{2m1^2}\right)\frac{1}{2m+1} = \frac{1}{2}
\end{aligned}$$

We have used Lemma 2.2 and the fact that $\left|\frac{p}{q}-\sqrt[3]{m}\right| < \frac{1}{bq^2}$.

Proof for $\sqrt[3]{m^2}$ is analogous. □

**Lemma 2.5.** *If the triplets* $(\frac{p_{n-1}}{q_{n-1}}, \xi_n, b_n)$ *and* $(\frac{P_{k-1}}{Q_{k-1}}, \eta_k, B_k)$ *are connected, then* $n$ *and* $k$ *are of different parity:* $(-1)^{k-1} = (-1)^n$.

*Proof.* It is enough to prove for $n$ is odd. From (3) it follows that

$$\frac{p_{n-1}}{q_{n-1}}\frac{P_{k-1}}{Q_{k-1}} = \sqrt[3]{m}\,\sqrt[3]{m^2} > \frac{p_{n-1}}{q_{n-1}}\sqrt[3]{m^2}.$$

So we have $\sqrt[3]{m^2} < \frac{P_{k-1}}{Q_{k-1}}$ and $k$ is even. □

**Theorem 2.6.** *Let* $(\frac{p_{n-1}}{q_{n-1}}, \xi_n, b_n)$ *and* $(\frac{P_{k-1}}{Q_{k-1}}, \eta_k, B_k)$ *be connected. Then there exist natural numbers* $r$, $s$, *such that* $rs = m$ *and*

$$-2r+2 \leq rb_n - sB_k \leq 2s-2. \tag{5}$$

*Proof.* Because convergents $\frac{p_{n-1}}{q_{n-1}}$, $\frac{P_{k-1}}{Q_{k-1}}$ are reduced, it follows from Definition 2.1 that

$$r := \frac{p_{n-1}}{Q_{k-1}}, \qquad s := \frac{P_{k-1}}{q_{n-1}} \tag{6}$$

are natural numbers.

Using (2) and (6) we get

$$\begin{aligned}
r\xi_n - s\eta_k &= -r\frac{p_{n-2}-q_{n-2}\sqrt[3]{m}}{p_{n-1}-q_{n-1}\sqrt[3]{m}} + s\frac{P_{k-2}-Q_{k-2}\sqrt[3]{m^2}}{P_{k-1}-Q_{k-1}\sqrt[3]{m^2}} \\
&= \frac{-rp_{n-2}+rq_{n-2}\sqrt[3]{m}}{rQ_{k-1}-q_{n-1}\sqrt[3]{m}} + \frac{sP_{k-2}-sQ_{k-2}\sqrt[3]{m^2}}{sq_{n-1}-Q_{k-1}\sqrt[3]{m^2}}
\end{aligned}$$

To obtain the same denominator, the second fraction is extended with $\sqrt[3]{m}$ and the factor $s$ divided out

$$\begin{aligned}
r\xi_n - s\eta_k &= \frac{-rp_{n-2}+rq_{n-2}\sqrt[3]{m}}{rQ_{k-1}-q_{n-1}\sqrt[3]{m}} + \frac{P_{k-2}\sqrt[3]{m}-mQ_{k-2}}{q_{n-1}\sqrt[3]{m}-rQ_{k-1}} \\
&= \frac{r(sQ_{k-2}-p_{n-2})-(P_{k-2}-rq_{n-2})\sqrt[3]{m}}{rQ_{k-1}-q_{n-1}\sqrt[3]{m}}
\end{aligned} \tag{7}$$

Let us solve the linear diophantine equation

$$p_{n-1}x - q_{n-1}y = (-1)^n r. \tag{8}$$

From (4) it follows that $p_{n-1}q_{n-2} - q_{n-1}p_{n-2} = (-1)^n$. Hence one of the solution of (8) is $(x_0, y_0) = (rq_{n-2}, rp_{n-2})$.

Let us prove that $(x_1, y_1) = (P_{k-2}, mQ_{k-2})$ is also the solution of (8) using (6), (4) and Lemma 2.5 we get

$$\begin{aligned} p_{n-1}x_1 - q_{n-1}y_1 &= p_{n-1}P_{k-2} - q_{n-1}mQ_{k-2} \\ &= rQ_{k-1}P_{k-2} - \frac{P_{k-1}}{s} rsQ_{k-2} \\ &= -r(P_{k-1}Q_{k-2} - Q_{k-1}P_{k-2}) \\ &= (-1)^{k-1}r \\ &= (-1)^n r \end{aligned}$$

Since $p_{n-1}$ and $q_{n-1}$ are coprime, the general solution of (8) can be written as

$$(x, y) = (x_0, y_0) + t(q_{n-1}, p_{n-1}), \quad t \in \mathbb{Z}.$$

Hence

$$(P_{k-2}, mQ_{k-2}) = (rq_{n-2}, rp_{n-2}) + t(q_{n-1}, p_{n-1})$$

and the parameter $t$ can be expressed in two forms

$$t = \frac{P_{k-2} - rq_{n-2}}{q_{n-1}} = \frac{mQ_{k-2} - rp_{n-2}}{p_{n-1}} \in \mathbb{Z}. \tag{9}$$

Using these two forms in the equation (7) it follows that $r\xi_n - s\eta_k = t$. On the other hand from (9) and (6) we can estimate

$$t = \frac{P_{k-2} - rq_{n-2}}{q_{n-1}} < \frac{P_{k-1} - rq_{n-2}}{q_{n-1}} = \frac{sq_{n-1} - rq_{n-2}}{q_{n-1}} < s$$

and

$$t = \frac{mQ_{k-2} - rp_{n-2}}{p_{n-1}} > \frac{mQ_{k-2} - rp_{n-1}}{p_{n-1}} = \frac{rsQ_{k-2} - r^2Q_{k-1}}{rQ_{k-1}} > -r.$$

Since $t \in \mathbb{Z}$ it follows

$$-r + 1 \le r\xi_n - s\eta_k \le s - 1. \tag{10}$$

We decompose the complete quotients into their integral and fractional parts:

$$\begin{aligned} \xi_n &= \lfloor \xi_n \rfloor + (\xi_n), & \frac{i}{r} < (\xi_n) < \frac{i+1}{r}, & \qquad 0 \le i \le r-1 \\ \eta_k &= \lfloor \eta_k \rfloor + (\eta_k), & \frac{j}{s} < (\eta_k) < \frac{j+1}{s}, & \qquad 0 \le j \le s-1 \end{aligned}$$

If we use this decomposition in the equation (10), we get

$$-r + 1 \le rb_n + i + \epsilon_1 - sB_k - j - \epsilon_2 \le s - 1$$

$$-2r + 2 + \epsilon_2 - \epsilon_1 \le rb_n - sB_k \le 2s - 2 + \epsilon_2 - \epsilon_1$$

where $0 < \epsilon_1, \epsilon_2 < 1$ and $\epsilon_2 - \epsilon_1$ is between -1 and 1. Because all other numbers are integers, we finaly get

$$-2r + 2 \le rb_n - sB_k \le 2s - 2.$$

□

**Corollary 2.7.** *If in the above Theorem 2.6 prime $m$ is taken, then the ratio between the connected partial quotients is roughly $m$ and the biggest one is at least $m$.*

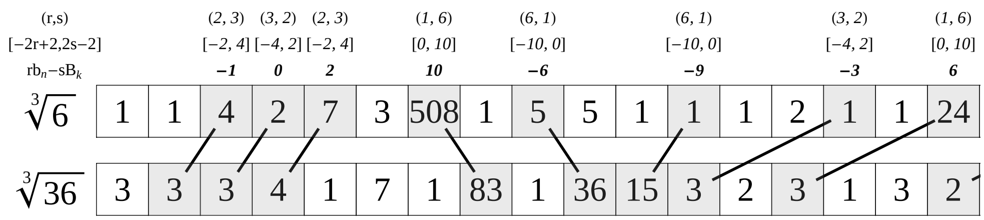


FIGURE 2. Continued fraction ladder of $(\sqrt[3]{6}, \sqrt[3]{36})$

**Lemma 2.8.** *In the case of consecutive connections the role of $r$, $s$ in Theorem 2.6 is interchanged (see Figure 2).*

*Proof.* From assumptions we have

$$\frac{p_{n-1}}{q_{n-1}}\frac{P_{k-1}}{Q_{k-1}} = m = rs \qquad r = \tfrac{p_{n-1}}{Q_{k-1}} \qquad s = \frac{P_{k-1}}{q_{n-1}}$$

$$\frac{p_{n-2}}{q_{n-2}}\frac{P_{k-2}}{Q_{k-2}} = m = r_1 s_1 \qquad r_1 = \tfrac{p_{n-2}}{Q_{k-2}} \qquad s_1 = \frac{P_{k-2}}{q_{n-2}}$$

We want to prove that $r_1 = s$ and $s_1 = r$.
From equation (4)

$$p_{n-1}q_{n-2} - p_{n-2}q_{n-1} = (-1)^n$$

we get

$$\frac{rQ_{k-1}P_{k-2}}{s_1} - \frac{r_1 Q_{k-2}P_{k-1}}{s} = (-1)^n$$

$$Q_{k-1}P_{k-2} - Q_{k-2}P_{k-1} = (-1)^n \frac{ss_1}{m}.$$

From equation (4) and Lemma 2.5 it follows $ss_1 = m$. □

**Lemma 2.9.** *If there are three consecutive connections between $b_{n-1}$, $b_n$, $b_{n+1}$ and $B_{k-1}$, $B_k$, $B_{k+1}$, then $rb_n - sB_k = 0$ for the middle one connection (see Figure 1 and 2).*

*Proof.* From assumptions we get using notation (6) from Theorem 2.6 for three consecutive indexes

$$r_{-1} = \frac{p_{n-2}}{Q_{k-2}},\ s_{-1} = \frac{P_{k-2}}{q_{n-2}},\ r = \frac{p_{n-1}}{Q_{k-1}},\ s = \frac{P_{k-1}}{q_{n-1}},\ r_1 = \frac{p_n}{Q_k},\ s_1 = \frac{P_k}{q_n}.$$

From Lemma 2.8 it follows

$$r_{-1} = s = r_1, \qquad s_{-1} = r = s_1.$$

We get

$$r\xi_n - s\eta_k = 0 \tag{11}$$

using

$$sQ_{k-2} - p_{n-2} = r_{-1}Q_{k-2} - p_{n-2} = 0,\ P_{k-2} - rq_{n-2} = P_{k-2} - s_{-1}q_{n-2} = 0$$

and (7). Similarly we have

$$r_1\xi_{n+1} - s_1\eta_{k+1} = 0. \tag{12}$$

In (12) we use definition of complete quotients (1) and get

$$\frac{s}{\xi_n - b_n} - \frac{r}{\eta_k - B_k} = 0.$$

After multiplying with denominators and using (11) we get the result. $\square$

**Remark 2.10.** Let us take the ladder of $(\sqrt[3]{2}, \sqrt[3]{4})$ with length 1000. In Figure 3 we can see the difference $n-k$ for positions of 665 ladder connections.

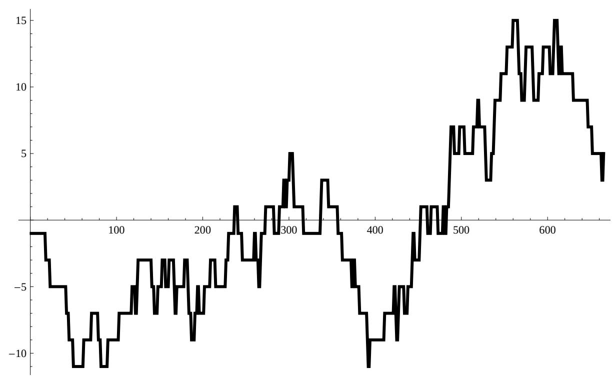


FIGURE 3. $n-k$ for ladder of $(\sqrt[3]{2}, \sqrt[3]{4})$ with length 1000

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